SOME BALANCED COMPLETE BLOCK DESIGNS

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ABSTRACT

Let $q = 6t \pm 1$, $v = 2q + 2$. The $(v/3)$ tri ples on v marks may be partitioned into q sets, each forming a BIBD of parameters $(v,3,2)$. Related results, some of them known, are also discussed briefly.

1. Notation and terminology

q is a natural integer with $(q,6)=1$, that is, of the form $6t \pm 1$. G is an additively written abelian group of order q, whose elements will be denoted by 0 and by lower case Latin letters, except x and v, with or without subscript. $H = G \oplus C(2)$; the generator of $C(2)$ is denoted by $\bar{0}$, and for $g \in G$, $g + \bar{0}$ will be written $\bar{g}$. V is a set consisting of the elements of H plus two additional marks x and y. $|V| = v = 2q + 2$. Terminology will follow rather closely that of Hall [2, mainly Sect. 15.3]. Thus, we shall attempt a mixed difference construction. Specifically, in Section 3 we shall select a set $S(0)$ of triples on the elements of V, being a BIBD of parameters $(v,3,2)$. This will form the first *layer* (corresponding to a *block* in the mixed difference procedure described in [2]). Then, for every $q \in G$, we shall obtain in Section 4 a new layer $S(g)$, by adding g to every element of H appearing in a triple of $S(0)$. These q BIBDs will have no triple in common and exhaust, among them, all $\binom{v}{3}$ triples on V. Taking such a set of q BIBDs as a single structure, with G acting as group of automorphisms, the designation of *complete* block design for it would appear justified. We shall then sketch, in Section 6, a similar partition of *triples* on a set into layers covering the *pairs once* (Steiner triples); next, of *pairs* into layers covering the *singletons once* (see (7.1), Matchings) or *twice* (see (7.2), Hamiltonian cycles), as was pointed out, in this context, by

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Simmons [5]. Lastly, a somewhat illegal example of partitioning the $(t+1)$-tuples of an infinite set into layers covering the t-tuples once.

2. Some more terminology and preliminary construction

Let τ denote the automorphism of G mapping each $g \in G$ into $-2g$. Then $0\tau = 0$ and since G can contain no element of order 3 (as $(q,3) = 1$), $g\tau \neq g$ for $g \neq 0$. Thus τ permutes the elements of $G^{\#}$ $(G^{\#} = G \setminus \{0\})$ in cycles. Call the ordered pair $(g, g\tau)$, $g \in G^{\#}$, an *arc* and $(-g\,, -g\tau)$ the *opposite* arc. There are thus $\frac{1}{2}(q-1)$ pairs of opposite arcs. Such pairs of arcs may lie on the same cycle (which is then of even length) or not (and then, sequences of such arcs form pairs of *opposite* cycles). Both situations may occur within the same group, as illustrated by $G = C(35)$, say.

Step 0. Of each pair of opposite arcs (on the same cycle or not) colour arbitrarily one red, one blue. (This may be done in $2^{(q-1)/2}$ ways.)

3. Construction of $S(0)$

Step 1*a*. Take each triple of different elements (a, b, c), $a, b, c \in G$, if $a + b + c = 0$.

Step 1*b*. For each triple thus formed, add 7 more, independently replacing a, b, c by $\bar{a}, \bar{b}, \bar{c}$ respectively.

Step 1*c*. For each *arc* (a,b) (that is, $b = -2a$), take the two triples $(a, b, \bar{a})$ and $(a, \bar{b}, \bar{a})$.

(So far, we have collected $\frac{4}{3}(q-1)(2q-1)$ triples, namely all the triples of *different* elements of H summing to 0 or $\bar{0}$; and, except for pairs of elements of H corresponding to pairs forming an *arc* in G, each pair of elements of H has been covered *twice*).

Step 2*a*. If (a,b) is a *red* arc, add the 4 triples (x, a, b), $(x, \bar{a}, \bar{b})$, $(y, a, \bar{b})$ $(y, \bar{a},, b)$.

Step 2*b*. If (a,b) is a blue arc, add the 4 triples (y, a, b), $(y, \bar{a}, \bar{b})$, $(x, a, \bar{b})$, $(x, \bar{a}, b)$.

Step 3. Add the 4 triples $(x, y, 0)$, $(x, y, \bar{0})$, $(x, 0, \bar{0})$, $(y, 0, \bar{0})$. Thus, each pair of elements of V has been covered exactly twice: the pairs originating from *arcs* have been covered a second time in Step 2 and for $g \in G^{\#}$, each pair (x, g), $(x, \bar{g})$, (y, g), $(y, \bar{g})$ has also been covered twice, once for g being the first element of an arc, once for its being the second. Step 3 completes this for $g = 0$ and for the pair (x,y), and we have indeed obtained a BIBD of type $(v,3,2)$.

4. Construction of $S(g)$

For each $g \in G$, $S(g)$ will consist of the following triples:

4.1. For each $(\alpha,\beta,\gamma) \in S(0)$, $\alpha,\beta,\gamma \in H$, $(\alpha+g, \beta+g, \gamma+g) \in S(g)$.

4.2. For each (x,α,β) or (y,α,β) in $S(0)$, $(x,\alpha+g, \beta+g)$ respectively $(y,\alpha+g, \beta+g)$ will be in $S(g)$.

4.3. $(x,y,g), (x,y,\bar{g}), (x,g,\bar{g}), (y,g,\bar{g}) \in S(g)$.

(It would be more in keeping with the notation in Hall [2] to write ∞_1 and ∞_2 for x and y).

This process of *translation* maps distinct elements, pairs, and triples of H again into distinct elements, pairs, and triples of H. Thus, since $S(0)$, obtained in Section 3, is a BIBD of type $(v,3,2)$, every $S(g)$ will be a design of the same type The total number of triples listed is q.

$$\frac{v(v-1)}{3} = \frac{v-2}{2} \cdot \frac{(v-1)v}{3} = \binom{v}{3}.$$

It remains therefore only to verify whether the sets $S(g)$ form in fact a *partition* of all the triples on V, that is, whether they are disjoint.

5. Check for common triples

Since $S(g_2)$ may be derived from $S(g_1)$ by the same operation of translation a $S(g_2 - g_1)$ from $S(0)$, it will be sufficient to examine $S(0)$ and $S(g)$ with $g \neq 0$ for common triples.

5.1. For $g \neq 0$, $S(g)$ contains none of the triples constructed in Step 1; for these sum to 0 or $\bar{0}$, while those of $S(g)$ sum to $3g$ or $3\bar{g}$ which cannot be 0 or $\bar{0}$ since $|G| = q$, $(q,3) = 1$.

5.2. Suppose now (a,b) is a *red* arc (Section 2) and $(x,a,b) \in S(0) \Pi S(g)$; put $a - g = c$, $b - g = d$; then by (4.2), $(x,c,d) \in S(0)$. Now (c,d) cannot be an arc if (a,b) is one, since $2c + d = 2a + b - 3g = -3g \neq 0$, by the argument above (5.1). Then (d,c) is an arc, that is, $2d + c = 2b + a - 3g = 0$ and, since $b = -2a$, this gives $-3a - 3g = 0$ or $g = -a$; therefore $c = 2a$, $d = -a$ so that $(d,c) = (-a, 2a)$ is the opposite arc to $(a, -2a) = (a,b)$. Now, by Step 0, (d,c) will be a blue arc; therefore, by Step 2b, $S(0)$ will contain (y,d,c) and following this, by

(4.2), $S(g)$ will contain (v, a, b) and not (x, a, b). By a similar argument (or rather, by 7 similar arguments) we may verify that none of the triples assigned to $S(0)$ by Step 2 are repeated elsewhere.

(Another formulation of what we have just proved would be as follows: Let e and f be two fixed elements of G. The unordered pair (e, f), under translation by the variable element $g \in G$, describes an *orbit* of unordered pairs $(e+g, f+g)$. This *orbit* will contain exactly two pairs that, when ordered, form *arcs*, say $(e+g_1, f+g_1)$ and $(f+g_2, e+g_2)$; moreover $f+g_2 = -(e+g_1)$, that is, in each orbit we find *one* pair of opposite arcs).

5.3. The case of the triples of Step 3 and, correspondingly (4.3), is trivial to verify.

Hence $S(0)$ is disjoint from $S(g)$ for $g \neq 0$ and consequently any two different $S(g)$s are disjoint. To sum up:

PROPOSITION 1. *If $v = 12t$ or $12t + 4$, the $\binom{v}{3}$ triples on v marks may be partitioned into $(v-2)/2$ sets, each forming a* BIBD *of parameters* $(v, v(v-1)/3, v-1, 3, 2)$.

REMARK. BIBDs of the form $(v, 3, 2)$ also exist for $v = 12t + 6$ and $v = 12t + 10$, and presumably an algorithm, similar to the one above, could be found to assemble $q = (v-2)/2$ such designs into a *complete* design on the $\binom{v}{3}$ triples; but the automorphism group would then be of even order, $q = 6t + 2$ or $6t + 4$ and some steps in the construction of $S(0)$ would have to be suitably changed. I have found some solutions for particular values of v, but a general construction would be desirable.*

In the next two sections, some more balanced complete block designs will be discussed, to emphasize the fact that the construction of Sections 3–4 is not an isolated instance.

6. Steiner triple systems

The following (arithmetical, rather than group-theoretical) property of groups has been proved in [3].

PROPOSITION 2. *Let G be a group of order q. Then the mapping $g \to -2g$ permutes the elements of $G^{\#}$ in cycles, each of length twice some odd number*

*For a recent class of examples of wide application in coding theory, see Ref. 6.

iff q divides some integer of the form $2^c - 1$, c odd; moreover, if this condition holds, $g \in G^{\#}$ and $-g$ will always be on the same cycle.

Example are $q = 7, 23, 31, 47, 49$.

To the elements of an *abelian* group of such order, let us adjoin two more marks, x and y, as in Section 1 above, to form a set V, $|V| = v = q + 2$. Modify Step 0 of Section 2 to read:

Step 0′. In every cycle, colour adjacent arcs alternately red and blue. The cycles being of even length, this colouring will be self-consistent, and each element of $G^{\#}$ will be on *one* arc of each colour; opposite arcs (half a cycle length, thus an odd number of steps, apart) will then automatically have different colours. If we now repeat the construction of $S(0)$ as in Section 3, ignoring triples with elements of $H - G$, each *pair* is covered *once* and a Steiner triple system results.

The *translation* of $S(0)$ to $S(g)$, as in Section 4, goes through unchanged, and the verification of disjointness, especially (5.2) above, remains valid since it rests on the fact that the only arc obtained from another by translation is the opposite one. We have thus the following proposition.

PROPOSITION 3. *If q divides some integer of the form $2^c - 1$, c odd, the $\binom{v}{3}$ triples on $v = q + 2$ marks may be partitioned into q sets (or layers), each forming a Steiner triple system.*

For other values of q, no such construction is known to the author*. A more detailed account appears in [4].

7. Simmons' uniform covers, and more examples

In [5], G. J. Simmons considers a question that may be termed *dual* to that of the existence of combinatorial t-designs. Let $n \geqq q \geqq r \geqq 1$; if a given q-set on n marks contains a given r-set, the r-set is said to *cover* the q-set. Each r-set thus covers $\binom{n-r}{n-q}$ q-sets; how many are required to cover all q-sets uniformly (an equal number, say λ, of times)? If less than $\binom{n}{r}$, the solution is termed *non-trivial*. As a by-product to his interesting results on this question, Simmons describes various classes of examples in which the r-sets may be partitioned into

* This has been recent progress on this question. See recent and forthcoming papers by R.H.F. Denniston, Alexander Rosa, L. Teirlinck and R.H. Wilson.

families of equal order, each forming a λ-cover of the q-sets. He stresses the equivalence between this description of the situation and the following one: replace each r-set by its complementary $(n-r)$-set and each q-set by its complementary $(n-q)$-set. Then the $(n-r)$-sets may be partitioned into *layers*, each layer covering (in the old, or t-design sense) λ times the $(n-q)$-sets.

Let us quote two typical instances.

7.1. MATCHINGS. Let $m = 2k+1$ and denote the elements of $C(m)$ (the cyclical group of order m) by $0, \pm 1, \pm 2, \cdots \pm k$. Adjoin an outside element x, and form a first layer $S(0)$ of the pairs $(x,0), (1,-1), (2,-2), \cdots, (k,-k)$. For $1 \leqq j \leqq k$, form $2k$ more layers $S(\pm j)$ by adding $+j$ or $-j$, modulo m, to each element of a pair, except x. Each pair appears exactly once, and each *singleton* is covered *once* in each layer. In graph theoretical language, we have coloured the edges of the complete graph on $2k+2$ vertices by $2k+1$ colours, the edges of each colour forming a *one-factor* or a *matching* of the graph. (So do the arcs of either colour, of the complete graph of the elements of $G^{\#}$, by step 0′ of Section 6) (see [5, Th. 9; a classical result]).

7.2. HAMILTONIAN CYCLES. Denote the elements of $C(m)$, $m = 2k$, by $0, \pm 1, \pm 2, \cdots \pm (k-1), k$. Adjoin an outside element x, as above, and form a first layer $S(0)$ of element pairs

$$(x,0), \quad (0,1), \quad (1,-1), \quad (-1,2)\cdots(k-1,1-k), \quad (1-k,k), \quad (k,x);$$

in other words, form the Hamiltonian cycle $(x,0,1,-1,\cdots k-1,1-k,k,x)$

Next, for $1 \leqq j \leqq k-1$, form $k-1$ more layers $S(j)$, by adding $j \in C(m)$ to every element, except x. Each layer contains every *singleton twice*, and between. them, all *pairs* are covered exactly once. We have thus decomposed the complete graph on $2k+1$ vertices into k edge-disjoint Hamiltonian cycles. (Simmons' construction [5, Th. 10] is more elegant, but more elaborate. The one here dates back to Kirkmann, refer to [1, Chap XIX, Example I], and would seem to indicate that the mixed difference procedure's applicability goes a bit farther than automorphisms of designs.) Note also some analogy between this example on the one hand, and the graph formed by the pairs associated with x, (or with y), in Steps 2a $-$2b of Section 3, and further in (4.2) on the other hand.

7.3. INFINITE t-DESIGNS. This last example is hardly legitimate, as it concerns an infinite set. However the similarity between it and the one in Section 6 on Steiner triples is readily apparent, and its structure, if anything, is simpler.

The set V consists, in this case, of the elements of an ordered abelian group G in which division by every integer $\leqq t+1$ is possible (for instance Q^+, the additive group of the rationals; but one may restrict denominators to power products of the primes $\leqq t+1$), and of t additional elements, x_1 to x_t.

$S(0)$ contains, in the first place, all that $(t+1)$-tuples of elements of G, such that:

(i) $g_1 < g_2 < g_3 < \cdots < g_t < g_{t+1}$,

(ii) $g_1 + g_2 + \cdots + g_{t+1} = 0$.

Condition (i) requires that, if the elements of the block are arranged in non-decreasing order, every difference $g_{i+1} - g_i$, $1 \leqq i \leqq t$, be positive. Next, if a set of elements of G satisfies (ii), but one or more difference $g_{i+1} - g_i$ is zero, we replace g_{i+1} by x_i and include this modified $(t+1)$-tuple in $S(0)$ as well. Thus, for instance, $S(0)$ contains the block $(0, x_1, x_2, \cdots, x_t)$. It is readily verified that $S(0)$ covers every t-tuple on V exactly once. Now, for every $g \in G$, we form $S(g)$ by adding g to every element of G in each block of $S(0)$. In this way, every $(t-1)$-tuple of V will appear exactly once. The verification is left to the reader.

I do not know of a finite example analogous to this for $t > 2$. $t = 1$ is (7.1) and for $t = 2$ we have the construction of Section 6. It would be very desirable to have finite examples for more values of t.

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